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On Optimal Guidance for Homing Missiles

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A simple class of differential games with terminal cost is treated. The results are applied to the problem of optimal guidance in the neighborhood of collision course. The evader responds ideally, while the pursuer has first-order dynamics. Both players have their bounded accelerations normal to the line of sight (LOS) as control variables. The optimal guidance law is simple and can be implemented in a closed form.

I. Introduction

THERE are two basic ways to formulate the problem of pursuit in the neighborhood of collision course. In the first we assume that both objects have *full state information,* but no information on the opponent's strategy. This leads to a two-person zero sum differential game. If a saddle point exists,[1-3] the value of the game is a guaranteed cost to each player no matter what strategy his opponent chooses. Thus, in the case of pursuit of an aircraft by a missile, if the latter uses his optimal strategy, the miss distance (at nominal time) will not be bigger than the value of the game for *any* (admissible) *aircraft's maneuver,* provided deviations from nominal collision course are small (see Ref. 4, Remark 6).

In the second formulation (e.g., see Refs. 5,6), it is assumed that the target has *no state information;* thus it performs a random telegraph maneuver (RTM).

From the rms point of view, RTM can be taken as a shaping filter driven by white noise. Thus stochastic optimal control can be used to minimize rms miss distance.

We may say therefore, that in the deterministic approach, the target operates rationally, while in the stochastic one it does not. Another difference between the two approaches is the possibility, in the deterministic approach, of considering hard bounds on control values, which to the knowledge of this author has not been done so far in the stochastic approach.

In a previous paper,[4] attention was given to optimal strategies in the neighborhood of collision course. In that paper it was assumed that both players respond ideally; they differ only in their maximum acceleration (normal to the LOS). In the present paper (Sec. II) we treat a simple class of differential games and shed some light on the case where a player's response is described by a time-invariant linear model (or by its equivalent transfer function). We construct the cost function using a simple Lyapunov-type function rather than the necessary conditions for saddle point. The latter, in fact, solves a more general class of games. Section III formulates in detail only the case where the evader E is ideal, while the pursuer P responds as a first-order system. Results (Sec. IV) are given from P's point of view.

II. A Simple Class of Differential Games

Consider the linear differential game

$$\dot{x}=A(t)x+B(t)u+C(t)v \qquad \text{dynamics} \tag{1a}$$

$$u\in U \quad v\in V \qquad \text{control constraints} \tag{1b}$$

$$J=\|Mx(T)\| \qquad \text{cost} \tag{1c}$$

where $x\in R^n$, $u\in R^m$, $v\in R^l$; $A,B,C,$ have proper dimensions; $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ are continuous, and $M\in R^{r\times n}$ is a constant matrix. Let $p(\cdot):R^{n+l}\to R^m$, $e(\cdot):R^{n+l}\to R^l$.

We wish to find an *optimal* strategy pair $\{p^*(\cdot), e^*(\cdot)\}$ among the family of *admissible* pairs $\{p(\cdot), e(\cdot)\}$ satisfying

i) $\{p(\cdot), e(\cdot)\}$ generates at least one solution $x(\cdot)$ of Eq. (1)
ii) $u(t)=p[x(t),t] \quad u\in U$
iii) $v(t)=e[x(t),t] \quad v\in V$

such that the following saddle point inequality holds:

$$J[x,t,p^*(\cdot),e(\cdot)]\le J[x,t,p^*(\cdot),e^*(\cdot)] \triangleq J^*(x,t)\le J[x,t,p(\cdot),e^*(\cdot)] \tag{2}$$

In order to simplify Eqs. (1), let

$\Phi(T,t)\triangleq$ the transition matrix of $\dot{x}=A(t)x$

i.e.,
$$\dot{\Phi}(T,t)=-\Phi(T,t)A(t) \tag{3a}$$

$$y\triangleq M\Phi(T,t)x \tag{3b}$$

$$X(T,t)\triangleq M\Phi(T,t)B(t) \tag{3c}$$

$$Y(T,t)\triangleq M\Phi(T,t)C(t) \tag{3d}$$

Using Eqs. (3) in Eqs. (1), one obtains

$$\dot{y}=X(T,t)u+Y(T,t)v \tag{4a}$$

$$u\in U \quad V\in V \tag{4b}$$

$$J=\|y(T)\| \tag{4c}$$

From Eqs. (4) and (2), it is clear that P (the minimizer) wishes to steer the state y (at $t=T$) as close to the origin as possible, while E (the maximizer) wishes the opposite. Since in Eqs. (4), $\{u,v\}$ operates directly on $\dot{y}$, we are *motivated* to choose a test function that measures the distance to the origin. We then observe the rate of change of this distance along a solution $y(\cdot)$. Let $D\triangleq\{(y,t): y\neq 0\}$ and let $W(\cdot): D\to R^1$ be given by

$$W(y,t)=\|y\| \tag{5}$$

Let

$$w(t)\triangleq Woy(t)\equiv\|y(t)\| \tag{6}$$

Using Eqs. (4) and (5),

$$\dot{w}(t)=\text{grad } W\cdot\dot{y}=\xi' X(T,t)u+\xi' Y(T,t)v \tag{7}$$

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where

$$\xi \triangleq \text{grad}\|y\| = y/\|y\|$$

We now restrict game (1) by the following:

Assumption 1

$$\alpha(\eta,t) \triangleq \underset{u\in U}{\text{Min}}\ \eta' X(T,t)u + \underset{v\in V}{\text{Max}}\ \eta' Y(T,t)v \equiv \alpha(t)$$

that is, $\alpha(\eta, t)$ is independent of η, for any $\eta \in \boldsymbol{R}^r$, $\|\eta\|=1$. This assumption holds in many interesting examples.† In particular it will be useful in obtaining simple guidance laws.

We now choose $\{p^*(\cdot), e^*(\cdot)\}$ by

$$\xi' X(T,t)p^* = \underset{u\in U}{\text{Min}}\ \xi' X(T,t)u \tag{8a}$$

$$\xi' Y(T,t)e^* = \underset{v\in V}{\text{Max}}\ \xi' Y(T,t)v \tag{8b}$$

Combining Eqs. (7) and (8), we find on D

$$\dot w(t)\,|_{p^*,e} \le \dot w(t)\,|_{p^*,e^*} \equiv \alpha(t) \le \dot w(t)\,|_{p,e^*} \tag{9}$$

Thus

$$\|y(T)\|\,|_{p^*,e} \le \|y_0\| + \int_{t_0}^{T} \alpha(t)\,\mathrm{d}t \le \|y(T)\|\,|_{p,e^*} \tag{10}$$

provided

$$\|y_0\| + \int_{t_0}^{t} \alpha(\tau)\,\mathrm{d}\tau \ge 0 \qquad t\in[t_0,T] \tag{11}$$

In this case the optimal cost becomes

$$J^*(y_0,t_0) = \|y_0\| + \int_{t_0}^{T} \alpha(t)\,\mathrm{d}t \tag{12}$$

Motivated by Eqs. (10-12), we construct the optimal cost function $V(\cdot)$: $\boldsymbol{R}^{r+1} \to \boldsymbol{R}^1$ defined by $V(y,t) = J^*(y_0, t_0)$, in case inequality (11) does not hold.

Let

$$t_s \triangleq \text{Max}\{\theta_s:\ \int_t^{\theta_s}\alpha(\tau)\,\mathrm{d}\tau = \inf_{\theta\in[t,T]} \int_t^{\theta}\alpha(\tau)\,\mathrm{d}\tau\} \tag{13}$$

Then

$$V(y,t) = \begin{cases} \int_{t_s}^{T}\alpha(\tau)\,\mathrm{d}\tau = \text{constant} & \forall (y,t)\in \bar D_1 \\ \|y\| + \int_t^T \alpha(\tau)\,\mathrm{d}\tau & \forall (y,t)\in D_2 - O \\ \beta' y + \int_t^T \alpha(\tau)\,\mathrm{d}\tau & \forall (y,t)\in O \end{cases} \tag{14}$$

where β is any constant unit vector, and

$$D_1 \triangleq \left\{(y,t):\ \|y\| + \int_t^{t_s}\alpha(\tau)\,\mathrm{d}\tau < 0;\ \ t<t_s\right\} \tag{15a}$$

$$D_2 \triangleq D_1^c \tag{15b}$$

$$O \triangleq \{(y,t):\ \ y=0\}\cap D_2 \tag{15c}$$

$\{p^*(\cdot), e^*(\cdot)\}$ is given by

arbitrary admissible pair $\qquad \forall(y,t)\in D_1 \qquad$ (16a)

Eq. (8) $\qquad \forall(y,t)\in D_2 - O \qquad$ (16b)

any $\lim_{(y,t)\to O} \{p^*(\cdot), e^*(\cdot)\}\,|_{D_2 - O} \qquad \forall(y,t)\in O \qquad$ (16c)

and the value of the game is

$$J(y,t) = \begin{cases} \int_{t_s}^{T}\alpha(\tau)\,\mathrm{d}\tau = \text{constant} & \forall (y,t)\in D_1 \\ \|y\| + \int_t^T \alpha(\tau)\,\mathrm{d}\tau & \forall (y,t)\in D_2 \end{cases} \tag{17}$$

In addition, it is easily verified that‡

i) $V(\cdot)$ is continuously differentiable with respect to the decomposition $\{D_1, O, D_2 - O\}$

ii)
$$\begin{aligned}\Delta(y,t,u,v) &\triangleq \text{grad}_y V\cdot \dot y + \partial V/\partial t \\ &= \xi' X(T,t)u + \xi' Y(T,t)v - \alpha(t)\quad \forall(y,t)\in D_2 \\ &= 0 \quad \forall(y,t)\in D_1\end{aligned}$$

satisfies

$$\begin{aligned}&\Delta(y,t,p^*(y,t),v) \le 0 \\ &\Delta[y,t,u,e^*(y,t)] \ge 0\end{aligned} \qquad \forall(y,t),\quad u\in U,\quad v\in V$$

where

$$\xi = \begin{cases}\text{any unit vector in } O \\ y/\|y\| \text{ in } D_2 - O\end{cases}$$

and

iii) $V(y,T) = \|y(T)\|$

We conclude by the following.

Theorem 1. Consider system (1) or, equivalently, (4). Suppose Assumption 1 holds. Then $\{p^*(\cdot), e^*(\cdot)\}$ given by Eq. (16) is a saddle point on $\boldsymbol{R}^n \times (-\infty, T)$. Moreover, the optimal cost is given by Eq. (17).

III. Motion of Two Objects in the Neighborhood of Collision Course

If two objects, P (pursuer) and E (evader), move with "small" deviations from a nominal collision course, the equations of motion normal to the line of sight (LOS) are.[1]

$$\dot x_1 = x_2$$

$$\dot x_2 = u_a + v_a$$

where $x_1 \triangleq e$ is the distance $P-E$ normal to the LOS; u_a is the actual pursuer's acceleration, normal to the LOS; and v_a is the actual evader's acceleration, normal to the LOS.

Assumption 2. i) The evader is ideal. ii) The pursuer's dynamics is given by a first-order linear system (with τ_p as a time constant).

Remark 1. In case the pursuer's dynamics is described by an nth order transfer function, M is still a row vector; thus Assumption 1 holds (see Appendix B1). The complete system is given below:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{\tau_p} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\tau_p} \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v \tag{18}$$

†See Appendix B.

‡See Appendix A.

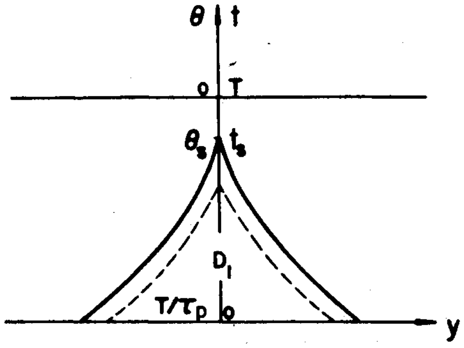

Fig. 1 (y,t)-space decomposition.

$$U=\{u:|u|\leq\rho_u\}\qquad V=\{v:|v|\leq\rho_v\}\quad \rho_u>\rho_v \tag{19}$$

$$J=|e(t)|=|Mx(T)|\quad\text{where}\quad M=[1\;\;0\;\;0] \tag{20}$$

The measured variable (output) is

$$z=\begin{bmatrix}\dot\sigma\\ x_3\end{bmatrix} \tag{21}$$

where, for small deviations,

$$\dot\sigma=\frac{\mathrm{d}}{\mathrm{d}t}\frac{e}{V_c(T-t)}=\frac{x_1}{V_c(T-t)^2}+\frac{x_2}{V_c(T-t)} \tag{22}$$

V_c = nominal closing speed

Remarks 2. The cost is the terminal miss distance normal to the LOS.

Remark 3. The meaning of the measured variable z is that eventually we shall need to express the saddle point strategies as functions of $\dot\sigma$ and x_3 only.

IV. Optimal Strategies

For system (18-22), we find

$$y=x_1+(T-t)x_2+\phi x_3=V_c(T-t)^2\dot\sigma+\phi x_3 \tag{23}$$

$$X(T,t)=\phi/\tau_p \tag{24a}$$

$$Y(T,t)=T-t \tag{24b}$$

where

$$\phi\triangleq\tau_p^2(e^{-\theta}+\theta-1)>0\quad\forall\theta>0 \tag{25a}$$

$$\theta\triangleq(T-t)/\tau_p \tag{25b}$$

$$\alpha(t)=-\rho_u(\phi/\tau_p)+\rho_v(T-t) \tag{26}$$

and t_s is the solution of

$$\left.\frac{e^{-\theta}+\theta-1}{\theta}\right|_{t_s}=\frac{\rho_v}{\rho_u} \tag{27}$$

A. General Solution

The decomposition of (y,t)-space (see Fig. 1) is as follows:

$$D_1=\left\{(y,t):\;|y|+\int_t^{t_s}\left[-\rho_u\frac{\phi}{\tau_p}+\rho_v(T-\tau)\right]\mathrm{d}\tau<0;t<t_s\right\} \tag{28a}$$

$$D_2=D_1^c \tag{28b}$$

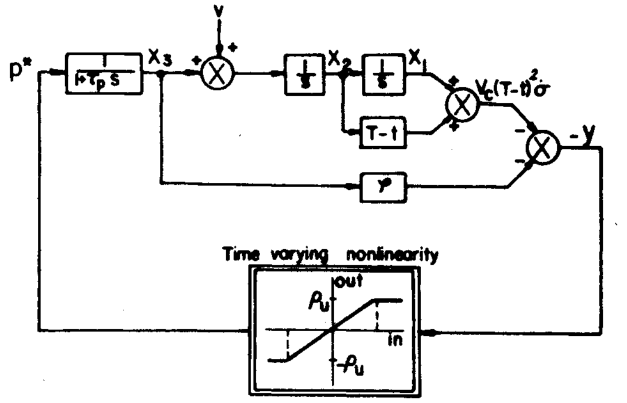

Fig. 2 Optimal pursuer's strategy.

$$O=\{(y,t):\;y=0\}\cap D_2 \tag{28c}$$

The optimal strategies are

$$\forall(y,t)\in D_1^c\;\begin{cases}p^*(y,t)=-\rho_u\operatorname{sgn}y & (29a)\\ e^*(y,t)=\rho_v\operatorname{sgn}y & (29b)\end{cases}$$

$\forall(y,t)\in D_1$, $\{p^*(\cdot),e^*(\cdot)\}$ is any admissible pair.

Note: $\operatorname{sgn}(o)\triangleq\pm1$, i.e. either + or −.

The value of the game

$$J(y,t)=\begin{cases}\rho_v\tau_p^2\theta_s\left[1-\dfrac{\theta_s}{2}\left(\dfrac{\rho_u}{\rho_v}-1\right)\right], & \forall(y,t)\in D_1\\[2ex] |y|+\displaystyle\int_t^T\left[-\rho_u\frac{\phi(\tau)}{\tau_p}+\rho_v(T-\tau)\right]\mathrm{d}\tau, & \forall(y,t)\in D_2\end{cases} \tag{30}$$

Remark 4. In D_1 we may *choose* a linear strategy with a slope (gain) given by

$$\text{slope}=\frac{1}{\displaystyle\int_t^{t_s}[(\phi/\tau_p)-(\rho_v/\rho_u)(T-\tau)]\mathrm{d}\tau} \tag{31}$$

Figure 2 shows the complete block diagram observed by the pursuer. Figure 3 shows the miss distance for $\rho_v\cong5g$ as a function of ρ_u/ρ_v with τ_p as a parameter.

B. A Practical Strategy

In order to obtain a simple expression in Eq. (31), we define

$$\begin{aligned}\tilde D_1&\triangleq\left\{(y,t):|y|+\int_t^T\left[-\rho_u\frac{\phi(\tau)}{\tau_p}+\rho_v(T-\tau)\right]\mathrm{d}\tau<0;\;t<\tilde t_s\right\}\\ &=\left\{(y,t):\;|y|+\rho_u\tau_p^2\theta^2\left[\frac{\psi}{\theta^2}-\frac{1}{2}\left(1-\frac{\rho_v}{\rho_u}\right)\right]<0;\;t<\tilde t_s\right\}\end{aligned} \tag{32}$$

where $\tilde t_s$ is the solution of

$$\psi/\theta^2|_{\tilde t_s}\triangleq(e^{-\theta}+\theta-1)/\theta^2|_{\tilde t_s}=½[1-\rho_v/\rho_u] \tag{33}$$

Remark 5. In Fig. 1, the dashed line represents $\tilde D_1$. Note that $\tilde D_1\subset D_1$. As before, the optimal strategies are

$$\forall(y,t)\in\tilde D_1^c\;\begin{cases}p^*(y,t)=-\rho_u\operatorname{sgn}y\\ e^*(y,t)=\rho_v\operatorname{sgn}y\end{cases} \tag{34a}$$

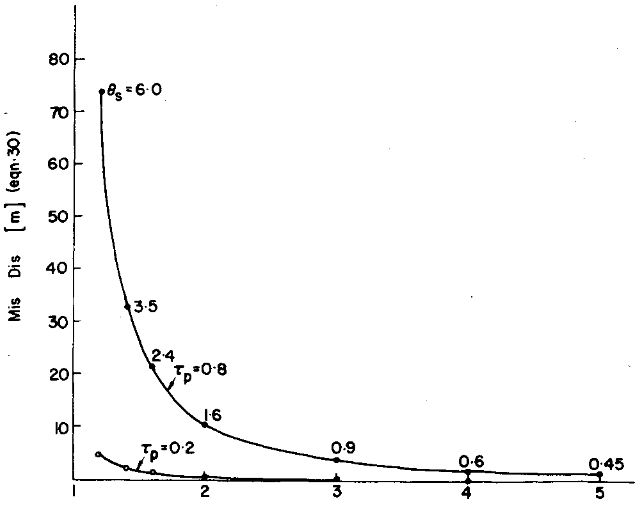

Fig. 3 Miss distance in D_I, $\rho_v = 50$ m/s².

$$\forall (y,t) \in \tilde{D}_I \quad \{p^*(\cdot), e^*(\cdot)\} \text{ is any admissible pair} \tag{34b}$$

If we *choose* a linear strategy in $\tilde{D}_I$, the slope becomes

$$\text{slope} = \frac{1}{[(1-\rho_v/\rho_u) - 2\psi/\theta^2]\tau_p^2\theta^2} \tag{35}$$

Thus, in the linear zone,

$$p^*(y,t) = -\text{slope}\cdot y = \frac{-2}{(1-\rho_v/\rho_u) - 2\psi/\theta^2}\left[V_c\dot{\sigma} + \frac{\psi}{\theta^2}x_3\right] \tag{36}$$

Let K_e, K_a be defined by

$$p^* = -K_e V_c \dot{\sigma} - K_a x_3 \tag{37}$$

Then

$$K_e = \frac{2}{(1-\rho_v/\rho_u) - 2\psi/\theta^2}$$

$$K_a = \frac{2\psi/\theta^2}{(1-\rho_v/\rho_u) - 2\psi/\theta^2} \tag{38}$$

Remark 6

$$\lim_{\theta\to 0}(\psi/\theta^2) = ½ \qquad \lim_{\tau_p\to 0}(\psi/\theta^2) = 0$$

Remark 7

$$\lim_{\tau_p\to 0} K_e = \frac{2}{1-\rho_v/\rho_u} \qquad \lim_{\tau_p\to 0} K_a = 0$$

Remark 8

$$K_e|_{\bar{t}_s} = K_a|_{\bar{t}_s} = \infty$$

Remark 9. If $(T-t)$ is sufficiently large, then at t,

$$K_e \cong \frac{2}{1-(\rho_v/\rho_u)} \qquad K_a \cong 0$$

C. The Limit Case $\tau_p = 0$[4]

In case the missile has an ideal dynamics, $\tau_p = 0$, it is found [using Eq. (13)] that $t_s = T$. Figure 4 shows the regions D_I, D_2.

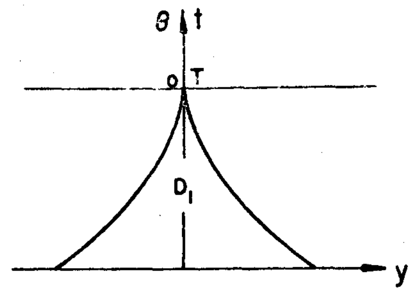

Fig. 4 (y,t)-space decomposition, $\tau_p = 0$.

In this case $p^*(\cdot)$ is bang-bang in D_2 and arbitrary admissible strategy in D_I. If a linear strategy is chosen in D_I, we obtain a *constant slope* such that

$$p^*(y,t) = -K_e V_c \dot{\sigma} \tag{39}$$

$$K_e = \frac{2}{1-(\rho_v/\rho_u)} \tag{40}$$

V. Conclusions

A game of two objects in the neighborhood of collision course was analyzed. Both players have hard bounds on their control values. The evader is ideal, while the pursuer has first-order dynamics. The optimal guidance law can be characterized as follows (Fig. 2):

1) The pursuer has to measure the rate of change $\dot{\sigma}$ of the LOS orientation and his acceleration x_3 normal to the LOS. Both variables are available for measurement (neglecting noise); thus the guidance law is easily implemented.

2) The control is bounded.

3) In $(\dot{\sigma}, x_3, T-t)$-space, there is a region D_I in which the optimal strategy is arbitrary; thus P may exercise in D_I a linear strategy with time-varying gains.

4) There exists an instant t_s at which these gains become infinite, and hereafter the optimal strategy is a pure bang-bang.

5) As P's time constant τ_p approaches zero, t_s approaches the nominal collision time T.

6) $\tau_p \neq 0$ implies that E can always guarantee himself a nonzero miss distance. The miss distance in D_I is presented in Fig. 3.

Appendix A

We present here a few results from Ref. 3 concerning integration and differential games.

Definition A1. A *denumerable decomposition* D of a set $X \subset \boldsymbol{R}^n$ is a denumerable collection of pairwise disjoint subsets whose union is X. We usually write $D = \{X_j : j \in J\}$, where J is a denumerable index set of the pairwise disjoint subsets.

Definition A2. Let X be a subset of $\boldsymbol{R}^n$ and D a denumerable decomposition of X. A real-valued continuous function V on X is said to be *continuously differentiable with respect to* D if, for $j \in J$, $V|X_j : X_j \to \boldsymbol{R}^1$ is continuously differentiable; that is, there exists a collection $\{(W_j, V_j) : j \in J\}$ such that W_j is an open set containing X_j, $V_j : W_j \to \boldsymbol{R}^1$ is continuously differentiable, and $V_j(x) = V(x)$ for $x \in X_j$.

Lemma A1. Let X be a subset of $\boldsymbol{R}^n$ and $D = \{X_j : j \in J\}$ be a denumerable decomposition of X. Let $\rho : [t_o, t_f] \to X$ be absolutely continuous and $h_o : [t_o, t_f] \to R^{-1}$ integrable. Let $V : X \to \boldsymbol{R}^1$ be continuous and continuously differentiable with respect to D. Let $T_j = \{t \in [t_o, t_f] : \rho(t) \in X_j\}$ for $j \in J$.

Suppose that for each $j \in J$

$$h_o(t) + \frac{d}{dt}(V_j o \rho)(t) \geq 0 \quad \text{a.e. in} \quad T_j$$

Then the function $g:[t_o, t_f] \to R^1$ defined by

$$g(t) = \int_{t_o}^{t} h_o(\tau)\mathrm{d}\tau + (Vo\rho)(t)$$

for $t \in [t_o, t_f]$ is monotone nondecreasing and absolutely continuous. The following result is the terminal cost version of Ref. 4, Corollary 1. Let $\dot{x} = f(x,t,u,v)$, $f(\cdot)$ is Borel measurable; $u \in U$, $v \in V$; and $J = m(x(T))$.

Theorem A1. If there exist a denumerable decomposition D of X and a continuous function $V:X \to R^1$ which is continuously differentiable with respect to D such that

i) grad $V_j(x,t) \cdot f(x,t,p^*,v) \leq 0 \quad \forall x \in X_j, \quad v \in V$

ii) grad $V_j(x,t) \cdot f(x,t,u,e^*) \geq 0 \quad \forall x \in X_j, \quad u \in U$

iii) $V(x,T) = m(x(T))$

Then $\{p^*(\cdot), e^*(\cdot)\}$ is an optimal strategry pair on $R^n \times (-\infty, T]$.

Appendix B

In this section we present two interesting cases in which Assumption 1 holds.

B1. $M \in R^{1 \times n} \Leftrightarrow y$ Is a Scalar

In this case $X(T,t)$ and $Y(T,t)$ are row vectors.

1):

$$U = \{u: \|u\| \leq \rho_u\}$$

$$V = \{v: \|v\| \leq \rho_v\}$$

Here, $\alpha(\xi,t)$ is attained at $\{\bar{p},\bar{e}\}$, where

$$\bar{p}(y,t) = -\rho_u \frac{X'(T,t)}{\|X(T,t)\|}\,\mathrm{sgn}y \qquad \mathrm{sgn}(o) \triangleq \pm 1$$

$$\bar{e}(y,t) = \rho_v \frac{Y'(T,t)}{\|Y(T,t)\|}\,\mathrm{sgn}y$$

and

$$\alpha(t) = -\rho_u \|X(T,t)\| + \rho_v \|Y(T,t)\|$$

2):

$$U = \{u: |u_i| \leq \rho_{ui}, \quad i = 1,2,\ldots,m\}$$

$$V = \{v: |v_i| \leq \rho_{vi}, \quad i = 1,2,\ldots,l\}$$

Here, $\alpha(\xi,t)$ is attained at $\{\bar{p},\bar{e}\}$, where

$$\bar{p}_i(y,t) = -\rho_{ui}\mathrm{sgn}(X_i y) \qquad \mathrm{sgn}(o) \triangleq \pm 1$$

$$\bar{e}_i(y,t) = \rho_{vi}\mathrm{sgn}(Y_i y)$$

and

$$\alpha(t) = -\sum_{i=1}^{m} \rho_{ui} |X_i(T,t)| + \sum_{i=1}^{l} \rho_{vi} |Y_i(T,t)|$$

B2. Orthogonal Systems

Definition B1. A linear system (4) is called *orthogonal* if

$$X(T,t) = \gamma_1(T,t)K_1(T,t)$$

$$Y(T,t) = \gamma_2(T,t)K_2(T,t)$$

where γ_i are scalers and K_i' are orthogonal matrices; that is $K_i K_i' = I$.

Let

$$U = \{u: \|u\| \leq \rho_u\}$$

$$V = \{v: \|v\| \leq \rho_v\}$$

Then $\alpha(\xi,t)$ is attained at $\{\bar{p},\bar{e}\}$, where

$$\bar{p}(y,t) = -\rho_u K_1'(T,t)\frac{y}{\|y\|}\mathrm{sgn}[\gamma_1(T,t)], \ \mathrm{sgn}(o) \triangleq \pm 1$$

$$\bar{e}(y,t) = \rho_v K_2'(T,t)\frac{y}{\|y\|}\mathrm{sgn}[\gamma_2(T,t)]$$

and

$$\alpha(t) = -\rho_u |\gamma_1(T,t)| + \rho_v |\gamma_2(T,t)|$$

Example. Let

$$x = \begin{bmatrix} x_p \\ x_e \end{bmatrix} \quad A = \begin{bmatrix} A_p & 0 \\ 0 & A_E \end{bmatrix} \quad B = \begin{bmatrix} \bar{B}_p \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 \\ \bar{B}_p \end{bmatrix} \qquad M = [M_p \quad -M_p]$$

where

$$A_p = \left[\begin{array}{c|c} 0 & I_{n-1} \\ \hline -a_o I & \cdots - a_{n-1} I \end{array}\right] \quad B_p = \left[\begin{array}{c} 0 \\ \hline I \end{array}\right] \quad M_p = [I \,|\, 0]$$

$$I \in R^{k+k}, I_{n-1} = \begin{bmatrix} I & & 0 \\ & \ddots & \\ 0 & & I \end{bmatrix}$$

Similar matrices exist for E.

It is easily verified that

$$X(T,t) = X(T-t) \triangleq X(\tau) = M_p \Phi_p(\tau) B_p$$

$$= I L_{(\tau)}^{-1}\left[\frac{1}{s^n + a_{n-1}s^{n-1} + \ldots + a_o}\right]$$

where $L_{(\tau)}^{-1}$ is the inverse Laplace transform operator and s is the Laplace variable. A similar relation holds for $Y(\tau)$. Clearly $X(\tau)$ and $Y(\tau)$ are orthogonal matrices.

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